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Finite Section Method for Linear Ordinary Differential Equations

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Sufficient conditions are given for the solution of a linear ordinary differential equation on the half line to be obtained as the limit of solutions of corresponding equations on finite intervals. Both the time-variant and the time-invariant case are considered, and in the latter case the sufficient conditions referred to above are also shown to be necessary. Included are applications to a differential equation that appears in linear transport theory, and to integral equations with semi-separable kernels. © 2000 Academic Press

1. INTRODUCTION

In this paper we study solutions of linear ordinary differential equations as limits of corresponding equations on a finite interval. To be more specific, we consider the equation

$$\begin{aligned} \dot{x}(t) - A(t)\,x(t) &= f(t), \qquad 0 \leqslant t < \infty, \\ Lx(0) &= 0, \end{aligned} \tag{1}$$

and a corresponding equation on the finite interval of the form

$$\begin{aligned} \dot{x}(t) - A(t)\,x(t) &= f(t), \qquad 0 \leqslant t \leqslant \tau, \\ Lx(0) + N(\tau)\,x(\tau) &= 0, \end{aligned} \tag{2}$$

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where $A(t)$ is a locally integrable $n \times n$ matrix function on $[0, \infty)$, and L and $N(\tau)$ are $n \times n$ matrices such that rank $L +$ rank $N(\tau) = n$. The problem is to obtain the solution of (1) as the limit for $\tau \to \infty$ of the solution of (2).

Throughout this paper we shall impose the following conditions on the function f and the coefficients and the boundary conditions of (1) and (2). We assume that the right hand side f in (1) is in $L_p^n[0, \infty)$, where $1 \leqslant p < \infty$ is fixed, and we require the solution to be in $L_p^n[0, \infty)$. In order for Eq. (1) to have a unique solution in $L_p^n[0, \infty)$ for each right hand side f in $L_p^n[0, \infty)$, we also assume that L is an *exponential dichotomy* of

$$\dot{x}(t) - A(t)\,x(t) = 0, \qquad 0 \leqslant t < \infty. \tag{3}$$

The latter means that L is a projection and that there exist positive real constants M and α such that

$$\begin{aligned} \|U(t)\,LU(s)^{-1}\| &\leqslant M\mathrm{e}^{-\alpha(t-s)}, \qquad 0 \leqslant s \leqslant t < \infty, \\ \|U(t)(I-L)\,U(s)^{-1}\| &\leqslant M\mathrm{e}^{-\alpha(s-t)}, \qquad 0 \leqslant t \leqslant s < \infty, \end{aligned} \tag{4}$$

where $U(t)$ is the fundamental matrix of (3). Furthermore, we shall restrict ourselves to the case when for τ sufficiently large Eq. (2) has a unique solution for each right hand side f in $L_p^n[0, \infty)$. The latter happens (see [7]) if and only if

$$\det(L + N(\tau)\,U(\tau)) \neq 0.$$

Given $f \in L_p^n[0, \infty)$, our aim is to approximate the unique solution x of (1) by the solution x_τ of (2) for $\tau \to \infty$. More precisely, we consider the problem of finding conditions guaranteeing that for $\tau \to \infty$ the solution x_τ of the Eq. (2) converges in L_p to the solution x of (1), where convergence of x_τ to x in L_p means that

$$\int_0^\tau \|x_\tau(t) - x(t)\|^p\,dt \to 0 \qquad (\tau \to \infty).$$

We shall say that for Eq. (1) the *finite section method with respect to the boundary value matrices* $\{N(\tau)\}$ *converges in* L_p if for each f in $L_p^n[0, \infty)$ the unique solution x of (1) in $L_p^n[0, \infty)$ is obtained as limit in L_p of the unique solution x_τ $(\tau \geqslant \tau_\circ)$ of Eq. (2). The first main result (Theorem 2.3) is that the latter happens whenever the boundary condition $N(\tau)$ satisfies

$$\sup_{\tau \geqslant \tau_\circ} \|U(\tau)(L + N(\tau)\,U(\tau))^{-1}\,N(\tau)\| < \infty.$$

This result is specified further for the time-invariant case when $A(t)$ does not depend on t. In this case we also assume that $N(t)$ does not depend on t. We show (see Theorem 3.2) that for this time-invariant case the sufficient condition for convergence of the finite section method mentioned above is also necessary.

The result for the time-invariant case is illustrated for a problem from linear transport theory. The problem asks to relate the solutions of the finite slab problem to the solutions of the half range problem. Here we restrict ourselves to the case when the scattering is in a finite number of directions only.

The results are also applied to integral equations

$$\phi(t)+\int_0^\infty k(t,s)\,\phi(s)\,ds=f(t), \qquad 0\leqslant t<\infty, \tag{5}$$

with a semi-separable kernel given by

$$k(t,s)=\begin{cases} C(t)\,U(t)\,LU(s)^{-1}\,B(s), & 0\leqslant s<t<\infty,\\ -C(t)\,U(t)(I-L)\,U(s)^{-1}\,B(s), & 0\leqslant t<s<\infty.\end{cases}$$

Here, as usual (cf. [5, Sect. III.3]), the finite section method means that the solutions of (5) are approximated by solutions of the equation

$$\phi(t)+\int_0^\tau k(t,s)\,\phi(s)\,ds=f(t), \qquad 0\leqslant t\leqslant\tau.$$

Our results for (5) extend and develop further those of [2].

The paper consists of five sections. The second section contains the main result for time-variant differential equations. Section 3 concerns the time-invariant case. The application to linear transport theory appears in Section 4. In Section 5 results on integral equations with semi-separable kernels are derived from the results on the differential equations.

2. A FINITE SECTION METHOD FOR DIFFERENTIAL EQUATIONS WITH DICHOTOMY

Throughout this section $U(t)$ is the fundamental matrix of the differential equation (3), i.e., $U(t)$ is absolutely continuous on finite intervals, $U(0)$ is the $n\times n$ identity matrix, and $(d/dt)\,U(t)=A(t)\,U(t)$ a.e. on $0\leqslant t<\infty$. Let L be a projection of $\mathbb{C}^n$. If L is an exponential dichotomy of $U(t)$, i.e., the inequalities (4) hold true, then L is also called an exponential dichotomy for Eq. (3). A dichotomy L (assuming that it exists) is not unique. In fact, only the image of L is determined by (4). This is the contents of the next proposition (see [4, pp. 16, 17]).

Proposition 2.1. *Let L be an exponential dichotomy of* (3), *and let $U(t)$ be the fundamental matrix of* (3). *Then for any p with $1 \leqslant p < \infty$*

$$\operatorname{Im} L = \{x \in \mathbb{C}^n \mid U(t)x \in L_p^n[0, \infty)\}. \tag{6}$$

Moreover, if L' is a projection of $\mathbb{C}^n$ such that $\operatorname{Im} L = \operatorname{Im} L'$, *then L' is also an exponential dichotomy of* (3).

In the time-invariant case, i.e., when $A(t) = A$ for each $t \geqslant 0$, Eq. (3) has an exponential dichotomy if and only if A has no eigenvalues on the imaginary axis, and in that case L is an exponential dichotomy if and only if $\operatorname{Im} L$ is the subspace of $\mathbb{C}^n$ spanned by the eigenvectors and generalized eigenvectors of A corresponding to eigenvalues in the left half plane.

Fix $f \in L_p^n[0, \infty)$ $(1 \leqslant p < \infty)$, and consider the equation

$$\begin{aligned} \dot{x}(t) - A(t)\, x(t) &= f(t), \qquad 0 \leqslant t \leqslant \infty, \\ Lx(0) &= 0, \end{aligned} \tag{7}$$

and a corresponding equation on the finite interval,

$$\begin{aligned} \dot{x}(t) - A(t)\, x(t) &= f(t), \qquad 0 \leqslant t < \tau, \\ Lx(0) + N(\tau)\, x(\tau) &= 0. \end{aligned} \tag{8}$$

Recall that throughout this paper we assume that $A(t)$ is a locally integrable $n \times n$ matrix function on $[0, \infty)$, that L is an exponential dichotomy for the equation $\dot{x}(t) = A(t)\, x(t)$, $t \geqslant 0$, and that $N(\tau)$ is an $n \times n$ matrix such that $\operatorname{rank} L + \operatorname{rank} N(\tau) = n$ for each $\tau > 0$. Also, $1 \leqslant p < \infty$ is fixed.

Theorem 2.2. *Assume that $L + N(\tau)\, U(\tau)$ is invertible for each $\tau \geqslant \tau_\circ$. If, in addition,*

$$\sup_{\tau \geqslant \tau_\circ} \| U(\tau)(L + N(\tau)\, U(\tau))^{-1}\, N(\tau) \| < \infty, \tag{9}$$

then for (7) *the finite section method relative to the boundary value matrices $\{N(\tau)\}$ converges in L_p.*

We shall derive Theorem 2.2 as a corollary of a slightly more general version in which we replace the right hand side of (8) by a function f_τ in $L_p^n[0, \tau)$ such that for $\tau \to \infty$ the function f_τ converges in L_p to f. We consider the equation

$$\begin{aligned} \dot{x}(t) - A(t)\, x(t) &= f_\tau(t), \qquad 0 \leqslant t \leqslant \tau, \\ Lx(0) + N(\tau)\, x(\tau) &= 0. \end{aligned} \tag{10}$$

As before, we assume that $\operatorname{rank} L + \operatorname{rank} N(\tau) = n$ for each $\tau > 0$.

THEOREM 2.3. *Assume that* $L+N(\tau)\,U(\tau)$ *is invertible for* $\tau \geqslant \tau_\circ$, *and that* f_τ *converges to* f *in* L_p. *If, in addition,*

$$\sup_{\tau \geqslant \tau_\circ} \|U(\tau)(L+N(\tau)\,U(\tau))^{-1}\,N(\tau)\| < \infty, \tag{11}$$

then the solution x_τ *of* (10) *converges to the solution* x *of* (7) *in* L_p

Proof. Part (a). It is well known [7, Sect. I.2] that (10) is uniquely solvable whenever $L+N(\tau)\,U(\tau)$ is invertible and in this case the solution x_τ of (10) is equal to $T_\tau f_\tau$, where T_τ is the integral operator on $L_p^n[0, \infty)$ given by

$$(T_\tau f_\tau)(t) = \int_0^\tau \gamma_\tau(t, s)\, f_\tau(s)\, ds, \qquad 0 \leqslant t \leqslant \tau, \tag{12}$$

with

$$\gamma_\tau(t, s) = \begin{cases} U(t)(I-P(\tau))\,U(s)^{-1}, & 0 \leqslant s < t \leqslant \tau, \\ -U(t)\,P(\tau)\,U(s)^{-1}, & 0 \leqslant t < s \leqslant \tau, \end{cases} \tag{13}$$

and $P(\tau) = (L+N(\tau)\,U(\tau))^{-1}\,N(\tau)\,U(\tau)$. Also (cf. [7, Proposition I.1.1]) $P(\tau)$ is a projection with $\operatorname{Im} P(\tau) = \operatorname{Ker} L$.

Next we derive an integral representation for x similar to the one we found for x_τ. From the variation of constants formula and the fact that $\|(I-L)\,U(s)^{-1}\| \leqslant M e^{-\alpha s}$ it follows that

$$0 = \lim_{t \to \infty} (I-L)\,U(t)^{-1}\,x(t) = \int_0^\infty (I-L)\,U(s)^{-1} f(s)\, ds + (I-L)\,x(0). \tag{14}$$

Since $x(0) \in \operatorname{Ker} L$, we can solve $x(0)$ from (14) and get that the solution $x(t)$ of (7) is given by $x = Tf$, where $T\colon L_p^n[0, \infty) \to L_p^n[0, \infty)$ is the integral operator defined by

$$(Tf)(t) = \int_0^\infty \gamma(t, s)\, f(s)\, ds, \qquad 0 \leqslant t < \infty, \tag{15}$$

with

$$\gamma(t, s) = \begin{cases} U(t)\,LU(s)^{-1}, & 0 \leqslant s < t < \infty, \\ -U(t)(I-L)\,U(s)^{-1}, & 0 \leqslant t < s < \infty. \end{cases} \tag{16}$$

The operator T is bounded (see, e.g., [8, Sect. I.2].)

Part (b). In the second part of the proof we show that x_τ converges in L_p to x. Denote the restriction of a function $f \in L_p^n[0, \infty)$ to $[0, \tau]$ by $R_\tau f$. We have to show that

$$\|T_\tau f_\tau - R_\tau Tf\|_{L_p^n[0, \tau]} \to 0 \qquad (\tau \to \infty). \tag{17}$$

For each $\tau > 0$ let E_τ be the canonical embedding of $L_p^n[0, \tau]$ in $L_p^n[0, \infty)$. Note that both operators R_τ and E_τ are of norm one. Furthermore, because f_τ converges in L_p to f, the quantity $\|f_\tau - R_\tau f\|_{L_p^n[0, \tau]}$ converges to zero for $\tau \to \infty$. Since the same holds true for $\|f - E_\tau R_\tau f\|_{L_p^n[0, \infty)}$, we have

$$\begin{aligned} &\|f - E_\tau f_\tau\|_{L_p^n[0, \infty)} \\ &\qquad \leqslant \|f - E_\tau R_\tau f\|_{L_p^n[0, \infty)} + \|E_\tau\| \; \|R_\tau f - f_\tau\|_{L_p^n[0, \tau]} \to 0 \qquad (\tau \to \infty). \end{aligned}$$

So it suffices to prove (17) with f replaced by $E_\tau f_\tau$. In other words, we have to show that

$$\begin{aligned} &\lim_{\tau \to \infty} \|R_\tau T E_\tau f_\tau - T_\tau f_\tau\|_{L_p^n[0, \tau]} \\ &\qquad = \lim_{\tau \to \infty} \left\| \int_0^\tau (\gamma(t, s) - \gamma_\tau(t, s)) \, f_\tau(s) \, ds \right\|_{L_p^n[0, \tau]} = 0. \end{aligned}$$

Because $\operatorname{Im} P(\tau) = \operatorname{Ker} L$, we have that $P(\tau) - I + L = P(\tau) L$ and $P(\tau) = (I - L) \, P(\tau)$. It follows that

$$\begin{aligned} &\gamma(t, s) - \gamma_\tau(t, s) \\ &\qquad = U(t)(P(\tau) - I + L) \, U(s)^{-1} \\ &\qquad = U(t)(I - L) \, P(\tau) \, L U(s)^{-1} \\ &\qquad = U(t)(I - L) \, U(\tau)^{-1} \, U(\tau)(L + N(\tau) \, U(\tau))^{-1} \, N(\tau) \, U(\tau) \, L U(s)^{-1}. \end{aligned} \tag{18}$$

Next we show that

$$\lim_{\tau \to \infty} \int_0^\tau \|U(\tau) \, L U(s)^{-1}\| \; \|f_\tau(s)\| \, ds = 0. \tag{19}$$

Since L is an exponential dichotomy for $U(t)$, it suffices to show that

$$\lim_{\tau\to\infty}\int_0^\tau M\mathrm{e}^{-\alpha(\tau-s)}\,\|f_\tau(s)\|\,ds=0.$$

Write F_t $(0<t\leqslant\infty)$ for the canonical embedding of $L_p[0,t)$ in $L_p(-\infty,\infty)$. We want to show that

$$\lim_{\tau\to\infty}\int_0^\infty M\mathrm{e}^{-\alpha s}(F_\tau\,\|f_\tau\|)(\tau-s)\,ds=0.$$

Since $F_\tau\,\|f_\tau\|\to F_\infty\,\|f\|$, this will follow from

$$\lim_{\tau\to\infty}\int_0^\infty M\mathrm{e}^{-\alpha s}(F_\infty\,\|f\|)(\tau-s)\,ds=0,$$

which is a known property of the convolution of $M\mathrm{e}^{-\alpha s}$ with $F_\infty\,\|f\|\in L_p(-\infty,\infty)$.

Next consider $\int_0^\tau\|U(t)(I-L)\,U(\tau)^{-1}\|^p\,dt$. Use again the fact that L is an exponential dichotomy to see that

$$\int_0^\tau\|U(t)(I-L)\,U(\tau)^{-1}\|^p\,dt\leqslant\int_0^\tau M^p\mathrm{e}^{-\alpha(\tau-t)p}\,dt=M^p\int_0^\tau\mathrm{e}^{-\alpha ps}\,ds,$$

which is a bounded function of τ. Since $U(\tau)(L+N(\tau)\,U(\tau))^{-1}\,N(\tau)$ is also bounded, we conclude that

$$\begin{aligned}\lim_{\tau\to\infty}&\left\|\int_0^\tau(\gamma(t,s)-\gamma_\tau(t,s))\,f_\tau(s)\,ds\right\|_{L_p^n[0,\tau]}\\&\leqslant\lim_{\tau\to\infty}M^p\int_0^\tau\mathrm{e}^{-\alpha ps}\,ds\,\|U(\tau)(L+N(\tau)\,U(\tau))^{-1}\,N(\tau)\|\\&\quad\times\int_0^\tau M\mathrm{e}^{-\alpha(\tau-s)}\,\|f_\tau(s)\|\,ds=0,\end{aligned}$$

as desired. ∎

We conclude this section with two remarks.

Remark 2.4. One can always choose the matrix function $N(\tau)$ in (8) such that the solution x_τ exists, is unique, and converges to the solution x of (7). The latter happens if $N(\tau) = (I - L)\, U(\tau)^{-1}$.

Indeed, with this choice of $N(\tau)$ we have for each $\tau > 0$ that rank $L +$ rank $N(\tau) = n$ and $L + N(\tau)\, U(\tau) = I$. Hence $L + N(\tau)\, U(\tau)$ is invertible, and condition (9) reduces to

$$\sup_{\tau \geqslant \tau_\circ} \| U(\tau)(I - L)\, U(\tau)^{-1} \| < \infty,$$

which holds true because L is an exponential dichotomy for $U(t)$.

However, the choice $N(\tau) = (I - L)\, U(\tau)^{-1}$ is not always the natural one. First, one would like to have $N(\tau)$ as simple as possible, and, for instance, not containing expressions such as $U(\tau)^{-1}$, which are not always easy to compute. Second, in the case when $A(t) = A$ is independent of t, one would prefer $N(\tau)$ to be independent of τ. In this time invariant case the above construction leads to $N(\tau) = (I - L)\, \mathrm{e}^{-\tau A}$, which is still dependent on τ. If, in addition, L and A commute, then one can choose $N(\tau) = I - L$, which is independent of τ. Third, in some problems the boundary condition is forced upon us by the problem and different from $(I - L)\, U(\tau)^{-1}$. In fact this happens in the example from transport theory considered in Section 4, cf. (34) and (36), and in the application to integral equations considered in Section 5 (see for example (46) and (47)).

The next remark concerns the condition of invertibility of $L + N(\tau)\, U(\tau)$ in Theorem 2.3. The fact that L is an exponential dichotomy implies that the half line equation (7) is uniquely solvable.

Remark 2.5. If L is an exponential dichotomy it does not follow that for large values of τ the finite interval equation (10) is uniquely solvable, not even if the operators $A(t)$ and $N(t)$ do not depend on t.

To see this choose

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N(\tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad A(t) = \begin{pmatrix} -1 & -\pi i & -2 \\ 0 & 1 & \pi i \\ 0 & \pi i & 1 \end{pmatrix}.$$

Then one computes that

$$L + N(\tau)\, \mathrm{e}^{\tau A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{\tau} \cos \pi\tau & -i\mathrm{e}^{\tau} \sin \pi\tau \\ \mathrm{e}^{-\tau} & 0 & \mathrm{e}^{-\tau} \end{pmatrix}.$$

This shows that in this case Eq. (10) is not uniquely solvable if $\tau = k + \frac{1}{2}$ with $k \in \mathbb{Z}$. On the other hand, the spectral projection corresponding to the eigenvalues in the right half plane is

$$P_A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which implies that L is an exponential dichotomy of (3).

3. TIME-INVARIANT CASE

Throughout this section A is an $n \times n$ matrix with no eigenvalues on the imaginary axis, and L_A is the spectral projection of A corresponding to the eigenvalues of A in the left half plane. This means that $\operatorname{Im} L_A$ is the subspace of $\mathbb{C}^n$ spanned by the eigenvectors and generalized eigenvectors corresponding to eigenvalues in the left half plane and $\operatorname{Ker} L_A$ is the subspace of $\mathbb{C}^n$ spanned by the (generalized) eigenvectors corresponding to eigenvalues in the right half plane. Since A has no eigenvalues on the imaginary axis, we know (see the paragraph after Proposition 2.1) that L_A is an exponential dichotomy of the equation

$$\dot{x}(t) - Ax(t) = 0, \qquad 0 \leqslant t < \infty. \tag{20}$$

Furthermore, throughout this section Q is a projection on $\mathbb{C}^n$ such that $\operatorname{Ker} Q \oplus \operatorname{Im} L_A = \mathbb{C}^n$.

We consider the differential equation

$$\begin{aligned} \dot{x}(t) - Ax(t) &= f(t), \qquad 0 \leqslant t < \infty, \\ Qx(0) &= 0. \end{aligned} \tag{21}$$

Here we assume that $f \in L_p^n[0, \infty)$. Equation (21) has a unique solution $x \in L_p^n[0, \infty)$. To see this let L be the projection with $\operatorname{Ker} L = \operatorname{Ker} Q$ and $\operatorname{Im} L = \operatorname{Im} L_A$. It follows from Proposition 2.1 that L is an exponential dichotomy of (20). Also $Qx(0) = 0$ if and only if $Lx(0) = 0$. Hence it follows from Part (b) in the proof of Theorem 2.3 that (21) has a unique solution in $L_p^n[0, \infty)$. Our aim is to obtain the unique solution of (21) as a limit for $\tau \to \infty$ of the solution x_τ of the equation

$$\begin{aligned} \dot{x}(t) - Ax(t) &= f(t), \qquad 0 \leqslant t < \tau, \\ Qx(0) + (I - Q)\, x(\tau) &= 0. \end{aligned} \tag{22}$$

Notice that the boundary condition in (22) is equivalent to the conditions $Qx(0)=0$ and $(I-Q)\,x(\tau)=0$ together. We shall say that for (21) *the finite section method relative to the boundary value matrix Q converges* if there exists a real number $\tau_\circ$ such that for each $\tau \geqslant \tau_\circ$ Eq. (22) has a unique solution, x_τ say, and x_τ converges in L_p to x.

THEOREM 3.1. *Let Q and L_A be as in the first paragraph of this section. For* (21) *the finite section method relative to the boundary value matrix Q converges if and only if $(I-Q)(I-L_A)+QL_A$ is invertible.*

As for the time-variant case, we derive this theorem as a corollary of a slightly more general result, in which we replace the right hand side in (21) by a function f_τ in $L_p^n[0, \tau]$ such that f_τ converges in L_p to f. More explicitly, we consider the equation

$$\begin{aligned} \dot{x}(t)-Ax(t)&=f_\tau(t), \qquad 0\leqslant t\leqslant \tau, \\ Qx(0)+(I-Q)\,x(\tau)&=0. \end{aligned} \tag{23}$$

THEOREM 3.2. *Let Q and L_A be as in the first paragraph of this section. Assume that $(I-Q)(I-L_A)+QL_A$ is invertible. Then there exists a number $\tau_\circ$ such that for each $\tau>\tau_\circ$ Eq.* (23) *has a unique solution x_τ. Furthermore, if f_τ converges in L_p to f, then x_τ converges in L_p to the solution x of* (21).

Conversely, assume that $(I-Q)(I-L_A)+QL_A$ is not invertible and that there exists a $\tau_\circ$ such that (23) *has a unique solution for each $\tau>\tau_\circ$. Then there exists a function $f\in L_p^n[0, \infty)$ such that for $f_\tau=f|_{[0,\tau]}$ the solution x_τ of* (23) *does not converge in L_p to the solution of* (21).

Proof. We split the proof into three parts. In the first part we derive equivalent forms for the invertibility of $(I-Q)(I-L_A)+QL_A$ and $Q+(I-Q)\,e^{\tau A}$. In the second part we assume that $(I-Q)(I-L_A)+QL_A$ is invertible and use Theorem 2.3 to prove the first half of the theorem; the third part concerns the converse statement.

Part (a). Let L be the projection of $\mathbb{C}^n$ such that $\operatorname{Im} L=\operatorname{Im} L_A$ and $\operatorname{Ker} L=\operatorname{Ker} Q$. Decompose $\mathbb{C}^n$ as $\operatorname{Im} L_A\oplus\operatorname{Ker} Q$. With respect to this decomposition we have the representations as operator matrices

$$L=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}, \qquad L_A=\begin{pmatrix} I & -R\\ 0 & 0\end{pmatrix}, \qquad Q=\begin{pmatrix} I & 0\\ -S & 0\end{pmatrix},$$

$$A=\begin{pmatrix} -A_{11} & A_{11}R+RA_{22}\\ 0 & A_{22}\end{pmatrix},$$

with the eigenvalues of A_{11} and A_{22} all in the right half plane. It follows that

$$e^{\tau A} = \begin{pmatrix} e^{-\tau A_{11}} & -e^{-\tau A_{11}} R + R e^{\tau A_{22}} \\ 0 & e^{\tau A_{22}} \end{pmatrix}.$$

A direct computation shows that

$$\begin{aligned}(I-Q)(I-L_A)+QL_A &= \begin{pmatrix} I & -R \\ -S & I+2SR \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I+SR \end{pmatrix}\begin{pmatrix} I & -R \\ 0 & I \end{pmatrix},\end{aligned}$$

and hence $(I-Q)(I-L_A)+QL_A$ is invertible if and only if $I+SR$ is invertible. Next one computes that

$$Q+(I-Q)\,e^{\tau A} = \begin{pmatrix} I & 0 \\ Se^{-\tau A_{11}} - S & (I+SR-SG(\tau))\,e^{\tau A_{22}} \end{pmatrix}, \tag{24}$$

where

$$G(\tau) = e^{-\tau A_{11}} R e^{-\tau A_{22}}. \tag{25}$$

From (24) we see that $Q+(I-Q)\,e^{\tau A}$ is invertible if and only if $I+SR-SG(\tau)$ is invertible. Notice that from (24) and the block matrix representations of Q and L it also follows that

$$L+(I-Q)\,e^{\tau A} = \begin{pmatrix} I & 0 \\ Se^{-\tau A_{11}} & (I+SR-SG(\tau))\,e^{\tau A_{22}} \end{pmatrix}. \tag{26}$$

Part (b). Assume that $(I-Q)(I-L_A)+QL_A$ is invertible. Then, by the first part of the proof, $I+SR$ is invertible. Since the eigenvalues of A_{11} and A_{22} are in the right half plane, we see from (25) that $G(\tau)\to 0$ for $\tau\to\infty$, and hence there exists a real number $\tau_\circ$ such that $I+SR-SG(\tau)$ is invertible for $\tau>\tau_\circ$. But then $Q+(I-Q)\,e^{\tau A}$ is invertible for $\tau>\tau_\circ$. It follows that for $\tau>\tau_\circ$ Eq. (23) has a unique solution x_τ.

Next, in addition, assume that f_τ converges in L_p to f. We shall apply Theorem 2.3 to show that x_τ converges in L_p to the solution x of (21). First recall that $\operatorname{Im} L = \operatorname{Im} L_A$. This shows that L is an exponential dichotomy of the equation $\dot{x}(t) - Ax(t) = 0$. Moreover, $\operatorname{Ker} L = \operatorname{Ker} Q$ and hence the boundary conditions at 0 in (23) and (21) can be rewritten as $Lx(0)=0$. The fundamental matrix of $\dot{x}(t)-Ax(t)=0$ is e^{tA}. In view of Theorem 2.3

it is now sufficient to show that $e^{\tau A}(L+(I-Q)\,e^{\tau A})^{-1}\,(I-Q)$ is a bounded function of τ. It follows from (26) that for $\tau>\tau_\circ$ we have

$$(L+(I-Q)\,e^{\tau A})^{-1}\,(I-Q)=\begin{pmatrix}0\\ e^{-\tau A_{22}}(I+SR-SG(\tau))^{-1}\end{pmatrix}(S\ \ I),$$

where $G(\tau)$ is given by (25), and thus that

$$e^{\tau A}(L+(I-Q)\,e^{\tau A})^{-1}\,(I-Q)=\begin{pmatrix}(R-G(\tau))(I+SR-SG(\tau))^{-1}\\ (I+SR-SG(\tau))^{-1}\end{pmatrix}(S\ \ I).$$

Now clearly

$$\begin{aligned}\lim_{\tau\to\infty}\ & e^{\tau A}(L+(I-Q)\,e^{\tau A})^{-1}\,(I-Q)\\ &=\begin{pmatrix}R(I+SR)^{-1}\\ (I+SR)^{-1}\end{pmatrix}(S\ \ I)\\ &=(I-L_A)(L+(I-Q)(I-L_A))^{-1}\,(I-Q).\end{aligned}$$

In particular, this shows that $e^{\tau A}(L+(I-Q)\,e^{\tau A})^{-1}\,(I-Q)$ is a bounded function of τ for $\tau\geqslant\tau_\circ$. We conclude that the solution of (23) converges to the solution of (21).

Part (c). Next assume that $(I-Q)(I-L_A)+QL_A$ is not invertible and that there exists a number $\tau_\circ$ such that (23) is uniquely solvable for $\tau>\tau_\circ$. By the results of Part (a) the latter means that $L+(I-Q)\,e^{\tau A}$ is invertible for $\tau>\tau_\circ$. Note that there exists a vector ϕ_0 such that $(I+SR)\,\phi_0=0$. Put

$$f(s)=\begin{pmatrix}-G(s)\,A_{22}(I+(e^{-\tau_\circ A_{22}}-I)^{-1}\,\chi_{(0,\,\tau_\circ)}(s))\,\phi_0\\ 0\end{pmatrix},$$

where $\chi_{(0,\,\tau_\circ)}(s)$ is the characteristic function of the interval $(0,\tau_\circ)$, and $G(\cdot)$ is given by (25). Then $f\in L_p^n[0,\infty)$. Let for this function f the solutions of (21) and (23) be x_f and $x_{\tau,f}$, respectively. We will show that

$$\liminf_{\tau\to\infty}\|x_f-x_{\tau,f}\|_{L_p^n(0,\,\tau)}>0. \tag{27}$$

Take $\tau>\tau_\circ$. Let $E_\tau f_\tau$ be the function on $[0,\infty)$ given by f_τ on $[0,\tau]$ and zero on (τ,∞), and let $x_{E_\tau f_\tau}$ be the solution of (21) with f replaced by $E_\tau f_\tau$.

Since $\|f - E_\tau f_\tau\|_{L_p^n[0,\infty)} \to 0$ $(\tau \to \infty)$, we conclude that $\|x_f - x_{E_\tau f_\tau}\|_{L_p^n[0,\infty)} \to 0$. Hence it is sufficient to show that

$$\liminf_{\tau \to \infty} \|x_{E_\tau f_\tau} - x_{\tau, f}\|_{L_p^n(0,\tau)} > 0. \tag{28}$$

By specifying (12), (15), and (18) for the time-invariant case considered here (with $E_\tau f_\tau$ replacing f) and using $E_\tau f_\tau(s) = 0$ for $s > \tau$, we see that for $0 \leqslant t \leqslant \tau$ we have

$$\begin{aligned} x_{\tau, f}(t) - x_{E_\tau f_\tau}(t) &= \int_0^\tau (\gamma_\tau(t,s) - \gamma(t,s))\, f(s)\, ds \\ &= \int_0^\tau \mathrm{e}^{tA}(I-L)(L+(I-Q)\,\mathrm{e}^{\tau A})^{-1}\,(I-Q)\,\mathrm{e}^{\tau A} L \mathrm{e}^{-sA} f(s)\, ds \\ &= \mathrm{e}^{tA}(I-L)(L+(I-Q)\,\mathrm{e}^{\tau A})^{-1}\,(I-Q)\,\mathrm{e}^{\tau A} L \int_0^\tau \mathrm{e}^{-sA} f(s)\, ds. \end{aligned}$$

Formula (26) yields that

$$\begin{aligned} &(I-L)(L+(I-Q)\,\mathrm{e}^{\tau A})^{-1}\,(I-Q)\,\mathrm{e}^{\tau A} L \\ &\quad = \begin{pmatrix} 0 & 0 \\ \mathrm{e}^{-\tau A_{22}}(I+SR-SG(\tau))^{-1}\, S\mathrm{e}^{-\tau A_{11}} & 0 \end{pmatrix}. \end{aligned}$$

Next, using our choice of f, we see that for $\tau > \tau_\circ$

$$L \int_0^\tau \mathrm{e}^{-sA} f(s)\, ds = \begin{pmatrix} \int_0^\tau \mathrm{e}^{sA_{11}} f_1(s)\, ds \\ 0 \end{pmatrix}, \tag{29}$$

where f_1 is the first block component of f. Now

$$\begin{aligned} \int_0^\tau \mathrm{e}^{sA_{11}} f_1(s)\, ds &= \int_0^\tau R\mathrm{e}^{-sA_{22}}(-A_{22})\,\phi_0\, ds \\ &\quad + \int_0^{\tau_\circ} R\mathrm{e}^{-sA_{22}}(-A_{22})\, ds (\mathrm{e}^{-\tau_\circ A_{22}} - I)^{-1}\,\phi_0 \\ &= R\mathrm{e}^{-\tau A_{22}}\phi_0. \end{aligned} \tag{30}$$

It follows that

$$(I-L)(L+(I-Q)\,\mathrm{e}^{\tau A})^{-1}\,(I-Q)\,\mathrm{e}^{\tau A}L\int_0^\tau \mathrm{e}^{-sA}f(s)\,ds$$
$$=\begin{pmatrix} 0 \\ \mathrm{e}^{-\tau A_{22}}(I+SR-SG(\tau))^{-1}\,SG(\tau)\,\phi_0 \end{pmatrix}.$$

Since $(I+SR)\,\phi_0=0$, we have that the second component in the right hand side is equal to $\mathrm{e}^{-\tau A_{22}}\phi_0$ and hence

$$(I-L)(L+(I-Q)\,\mathrm{e}^{\tau A})^{-1}\,(I-Q)\,\mathrm{e}^{\tau A}L\int_0^\tau \mathrm{e}^{-sA}f(s)\,ds=\begin{pmatrix} 0 \\ \mathrm{e}^{-\tau A_{22}}\phi_0 \end{pmatrix}.$$

Therefore

$$x_{\tau,f}(t)-x_{E_\tau f_\tau}(t)=\mathrm{e}^{tA}\begin{pmatrix} 0 \\ \mathrm{e}^{-\tau A_{22}}\phi_0 \end{pmatrix}.$$

The second component of the latter expression is $\mathrm{e}^{(t-\tau)A_{22}}\phi_0$. The norm in $L_p(0,\tau)$ of this function is equal to the norm of $\mathrm{e}^{-tA_{22}}\phi_0$. Hence for $\tau\to\infty$ this quantity converges to the norm of $\mathrm{e}^{-tA_{22}}\phi_0$ in $L_p(0,\infty)$, which is non-zero. In particular, this implies (28). ∎

4. AN EXAMPLE FROM TRANSPORT THEORY

In this section we consider the boundary value problem

$$\begin{aligned} &T\dot{\psi}(t)=-\psi(t)+F\psi(t), \qquad 0\leqslant t<\infty,\\ &Q\psi(0)=x_+, \end{aligned} \tag{31}$$

where T and F are selfadjoint $n\times n$ matrices, T invertible, and Q is the spectral projection of T corresponding to the positive eigenvalues of T. We require solutions to be in $L_p^n[0,\infty)$. The initial value x_+ is in $\operatorname{Im} Q$. Our aim is to obtain the solutions of (31) as limits for $\tau\to\infty$ of the solutions of

$$\begin{aligned} &T\dot{\psi}(t)=-\psi(t)+F\psi(t), \qquad\qquad 0\leqslant t\leqslant\tau,\\ &Q\psi(0)=x_+, \qquad (I-Q)\,\psi(\tau)=x_-(\tau), \end{aligned} \tag{32}$$

where we assume that for each $\tau>\tau_\circ>0$ the boundary value $x_-(\tau)\in\operatorname{Ker} Q$. Equations (31) and (32) appear in linear transport theory if one assumes that the scattering is in a finite number of directions. (See Sections XIII.9

and XIX.7 in [6] for further details.) Notice that Eqs. (31) and (32) differ from their counterparts in Section 3 by the fact that the boundary values x_+ and $x_-(\tau)$ may be nonzero. By applying Theorem 3.2 we obtain the following result.

THEOREM 4.1. *Assume that $T^{-1}(I-F)$ has no eigenvalues on the imaginary axis, and let $L^\times$ be the spectral projection of $T^{-1}(I-F)$ corresponding to the eigenvalues in the right half plane. If $QL^\times+(I-Q)(I-L^\times)$ is invertible, then for each function $x_-(\tau)$ with $\lim_{\tau\to\infty} x_-(\tau)=0$, the solutions ψ_τ of (32) converge in L_p to the solution ψ of (31).*

Proof. With $A=-T^{-1}$, $B=-T^{-1}F$, and $A^\times=A-B=-T^{-1}(I-F)$ the equation (31) becomes

$$\begin{gathered}\dot\psi(t)-A^\times\psi(t)=0, \qquad 0\leqslant t<\infty,\\ Q\psi(0)=x_+.\end{gathered}\tag{33}$$

Put $f(t)=e^{tA}x_+$. Since Q is an exponential dichotomy of e^{tA}, we have that $f\in L_p^n[0,\infty)$. One computes that $\dot f(t)-A^\times f(t)=Ae^{tA}x_+-A^\times e^{tA}x_+=Bf(t)$. Moreover, $Qf(0)=x_+$. So ψ is a solution of (33) if and only if $\rho=\psi-f$ is a solution of

$$\begin{gathered}\dot\rho(t)-A^\times\rho(t)=-Bf(t), \qquad 0\leqslant t<\infty,\\ Q\rho(0)=0.\end{gathered}\tag{34}$$

Notice that $L^\times$ is an exponential dichotomy of $\dot\psi(t)-A^\times\psi(t)=0$.

Next we rewrite (32) as

$$\begin{gathered}\dot\psi(t)-A^\times\psi(t)=0, \qquad\qquad 0\leqslant t\leqslant\tau,\\ Q\psi(0)=x_+, \qquad (I-Q)\,\psi(\tau)=x_-(\tau),\end{gathered}\tag{35}$$

Put $f_\tau(t)=e^{tA}x_++e^{-(\tau-t)A}x_-(\tau)$. Then one computes that $\dot f_\tau(t)-A^\times f_\tau(t)=Bf_\tau(t)$ and that $Qf_\tau(0)=x_+$ and $(I-Q)\,f_\tau(\tau)=x_-(\tau)$. We find that ψ_τ is a solution of (35) if and only if $\rho_\tau=\psi_\tau-f_\tau$ is a solution of

$$\begin{gathered}\dot\rho(t)-A^\times\rho(t)=-Bf_\tau(t), \qquad\qquad 0\leqslant t\leqslant\tau,\\ Q\rho(0)=0, \qquad (I-Q)\,\rho(\tau)=0.\end{gathered}\tag{36}$$

Now we are in a position to apply Theorem 3.2 to Eqs. (34) and (36). Indeed read $A^\times$ in the place of A, replace L_A by $L^\times$, the function f by $-Bf$, and finally let $-Bf_\tau$ replace f_τ. Since the operator $QL^\times+(I-Q)(I-L^\times)$ is invertible, we may conclude that ρ_τ converges in L_p to ρ. This implies that the solutions ψ_τ of (32) converges in L_p to the solution of (31). ∎

Notice that in (36) the boundary condition at τ is given by the problem and differs from the "canonical" choice in Remark 2.4. Indeed, the condition $(I-Q)\,\rho(\tau)=0$ is different from the condition $(I-Q)\,\mathrm{e}^{-\tau A^\times}\rho(\tau)=0$, which is the one suggested by Remark 2.4 for this problem.

The next proposition shows that under a certain additional condition the invertibility of the operator $QL^\times+(I-Q)(I-L^\times)$ in Theorem 4.1 is automatically fulfilled.

PROPOSITION 4.2. *Let Q and $L^\times$ be as in Theorem* 4.1. *If $I-F$ is positive definite, then $QL^\times+(I-Q)(I-L^\times)$ is invertible.*

Proof. Put $A^\times=-T^{-1}(I-F)$ and $H=-T$. Then H is selfadjoint and $I-F=HA^\times$. From $\langle HA^\times x, x\rangle=\langle (I-F)\,x, x\rangle>0$ for each nonzero $x\in\mathbb{C}^n$, it follows that $HA^\times$ is positive. In other words, the operator $A^\times$ is H-positive. Since $L^\times$ is the spectral projection of $A^\times$ corresponding to the eigenvalues in the left half plane, the fact that $A^\times$ is H-positive implies that $\langle Hv, v\rangle\leqslant 0$ for each $v\in\operatorname{Im} L^\times$ (cf. [9, Theorem I.3.15]).

To prove the invertibility of $QL^\times+(I-Q)(I-L^\times)$ it suffices to show that this operator has a trivial kernel. Let $(QL^\times+(I-Q)(I-L^\times))\,x=0$. We have to prove that $x=0$. Put $y=L^\times x$ and $z=(I-L^\times)x$. Then $Qy+(I-Q)\,z=0$, and hence $Qy=0$ and $(I-Q)\,z=0$. Since Q is the spectral projection of H corresponding to the negative eigenvalues of H, we conclude that either $y=0$ or $\langle Hy, y\rangle>0$. On the other hand $y\in\operatorname{Im} L^\times$, and thus $\langle Hy, y\rangle\leqslant 0$ by the results mentioned in the previous paragraph. Hence $y=0$. Similarly one proves that $z=0$. We conclude that $x=0$. ∎

We express our gratitude to A.C.M. Ran for pointing out the above result and its proof. The condition that $I-F$ is positive definite in relation to the convergence of a finite section method appeared earlier in a slightly different context in [10].

5. INTEGRAL EQUATIONS WITH SEMI-SEPARABLE KERNELS

In this section we give the application to integral equations with semi-separable kernels. Throughout this section the projection L is an exponential dichotomy for the differential equation $\dot{x}(t)-A(t)\,x(t)=0$ on $0\leqslant t<\infty$, and $U(t)$ is the fundamental matrix of this differential equation. We consider the integral equation

$$\phi(t)+\int_0^\infty k(t,s)\,\phi(s)\,ds=f(t),\qquad 0\leqslant t<\infty, \tag{37}$$

where

$$k(t,s)=\begin{cases} C(t)\,U(t)\,LU(s)^{-1}\,B(s), & 0\leqslant s<t<\infty, \\ -C(t)\,U(t)(I-L)\,U(s)^{-1}\,B(s), & 0\leqslant t<s<\infty. \end{cases} \tag{38}$$

Here $B(t)$ is an $n\times m$ matrix and $C(t)$ is an $m\times n$ matrix, and the entries of both matrices are bounded measurable functions on the half line $[0,\infty)$. We require both the right hand side and the solution of (37) to be functions in $L_p^m[0,\infty)$. Our aim is to get the solution of (37) as a limit for $\tau\to\infty$ of the solution of the corresponding equation on the interval $[0,\tau]$.

In our analysis the matrix function $A^\times(t)=A(t)-B(t)\,C(t)$ will play an important role. Since the entries of both $B(t)$ and $C(t)$ are assumed to be bounded measurable functions on $[0,\infty)$, the matrix function $A^\times(t)$ is again locally integrable on $0\leqslant t<\infty$, and hence the differential equation

$$\dot{x}(t)-A^\times(t)\,x(t)=0, \qquad 0\leqslant t<\infty, \tag{39}$$

has a well-defined fundamental matrix, which we shall denote by $U^\times(t)$.

We shall say that *the finite section method for the integral equation* (37) converges if there exists a number $\tau_\circ$ such that for every $f\in L_p^m[0,\infty)$ and each $\tau\geqslant\tau_\circ$ the integral equation

$$\phi(t)+\int_0^\tau k(t,s)\,\phi(s)\,ds=f(t), \qquad 0\leqslant t<\tau, \tag{40}$$

has a unique solution $\phi_\tau\in L_p^m[0,\tau]$, which converges in L_p to the solution ϕ of (37).

This section contains two subsections. The first concerns the general case and in the second we treat the time-invariant case, i.e., the case when A, B, and C do not depend on t.

5.1. *The General Case*

The next theorem is the counterpart of Theorem 2.2 for integral operators.

THEOREM 5.1. *Let $k(t,s)$ be given by* (38). *Put $A^\times(t)=A(t)-B(t)\,C(t)$, and assume that the fundamental matrix $U^\times(t)$ of* (39) *has an exponential dichotomy $L^\times$ such that* $\operatorname{Im} L^\times\oplus\operatorname{Ker} L=\mathbb{C}^n$. *Assume that there exists a number $\tau_\circ$ such that the matrix function $L+(I-L)\,U(\tau)^{-1}\,U^\times(\tau)$ is invertible for $\tau>\tau_\circ$. If*

$$\sup_{\tau>\tau_\circ}\|U^\times(\tau)(L+(I-L)\,U(\tau)^{-1}\,U^\times(\tau))^{-1}\,(I-L)\,U(\tau)^{-1}\|<\infty, \tag{41}$$

then the finite section method for (37) *converges.*

Proof. According to [8, Theorem II.2.2], a function $\phi \in L_p^m[0, \infty)$ is a solution of (37) if and only if there exists a (unique) function $\rho \in L_p^n[0, \infty)$ such that with input $u = \phi$ the system

$$\begin{aligned} \dot{\rho}(t) &= A(t)\,\rho(t) + B(t)\,u(t), \qquad && 0 \leqslant t < \infty, \\ y(t) &= C(t)\,\rho(t) + u(t), && 0 \leqslant t < \infty, \\ L\rho(0) &= 0 \end{aligned} \tag{42}$$

has output $y = f$. Hence to solve (37) one inverts the system (42), i.e., one passes to the inverse system:

$$\begin{aligned} \dot{\rho}(t) &= A^{\times}(t)\,\rho(t) + B(t)\,y(t), \qquad && 0 \leqslant t < \infty, \\ u(t) &= -C(t)\,\rho(t) + y(t), && 0 \leqslant t < \infty, \\ L\rho(0) &= 0. \end{aligned} \tag{43}$$

For the latter system we know that if the input $y = f$, then the output $u = \phi$. However, we want that $\rho \in L_p^n[0, \infty)$. Therefore we put Π to be the projection of $\mathbb{C}^n$ such that $\operatorname{Im} \Pi = \operatorname{Im} L^{\times}$ and $\operatorname{Ker} \Pi = \operatorname{Ker} L$. Then on the one hand Π is an exponential dichotomy of $U^{\times}(t)$ and on the other hand the boundary condition of (43) is equivalent to $\Pi\rho(0) = 0$. So indeed (43) has a unique solution $\rho \in L_p^n[0, \infty)$. (See Part (b) in the proof of Theorem 2.3.)

Next we consider Eq. (40). According to [7, Theorem 2.1], a function $\phi_\tau \in L_p^m[0, \tau]$ is a solution of (40) if and only if there exists a (unique) function $\rho_\tau \in L_p^n[0, \tau]$ such that with input $u = \phi_\tau$ the system

$$\begin{aligned} \dot{\rho}(t) &= A(t)\,\rho(t) + B(t)\,u(t), \qquad && 0 \leqslant t \leqslant \tau, \\ y(t) &= C(t)\,\rho(t) + u(t), && 0 \leqslant t \leqslant \tau, \\ L\rho(0) &+ (I - L)\,U(\tau)^{-1}\,\rho(\tau) = 0 \end{aligned} \tag{44}$$

has output $y = f$. In order to solve (40) one inverts the system (44) to get

$$\begin{aligned} \dot{\rho}(t) &= A^{\times}(t)\,\rho(t) + B(t)\,y(t), \qquad && 0 \leqslant t \leqslant \tau, \\ u(t) &= -C(t)\,\rho(t) + y(t), && 0 \leqslant t \leqslant \tau, \\ L\rho(0) &+ (I - L)\,U(\tau)^{-1}\rho(\tau) = 0, \end{aligned} \tag{45}$$

which has the output $u = \phi$ if the input $y = f$. Since $S(\tau) = L + (I - L)\,U(\tau)^{-1} \times U^{\times}(\tau)$ is invertible, it follows that the system (45) is uniquely solvable. (See Part (a) in the proof of Theorem 2.3.) Again the first boundary condition can be written as $\Pi\rho(0) = 0$.

We intend to apply Theorem 2.3 to the differential equations

$$\begin{aligned} &\dot{\rho}(t) = A^{\times}(t)\,\rho(t) + B(t)\,f(t), \qquad 0 \leqslant t < \infty,\\ &\Pi\rho(0) = 0, \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\dot{\rho}(t) = A^{\times}(t)\,\rho(t) + B(t)\,f(t), \qquad\qquad 0 \leqslant t \leqslant \tau,\\ &\Pi\rho(0) = 0, \qquad (I-L)\,U(\tau)^{-1}\rho(\tau) = 0, \end{aligned} \tag{47}$$

in order to conclude that ρ_τ converges in L_p to ρ. Therefore it remains to show that

$$\sup_{\tau > \tau_\circ} \| U^{\times}(\tau)(\Pi + (I-L)\,U(\tau)^{-1}\,U^{\times}(\tau))^{-1}\,(I-L)\,U(\tau)^{-1} \| < \infty. \tag{48}$$

Now use that $(\Pi + I - L)(I-L) = (I-L)$ and $(\Pi + I - L)\,L = \Pi$ to verify

$$\begin{aligned} &U^{\times}(\tau)(L + (I-L)\,U(\tau)^{-1}U^{\times}(\tau))^{-1}\,(I-L)\,U(\tau)^{-1}\\ &\quad = (LU^{\times}(\tau)^{-1} + (I-L)\,U(\tau)^{-1})^{-1}\,(I-L)\,U(\tau)^{-1}\\ &\quad = (LU^{\times}(\tau)^{-1} + (I-L)\,U(\tau)^{-1})^{-1}\,(\Pi + I - L)^{-1}\,(I-L)\,U(\tau)^{-1}\\ &\quad = ((\Pi + I - L)(LU^{\times}(\tau)^{-1} + (I-L)\,U(\tau)^{-1}))^{-1}\,(I-L)\,U(\tau)^{-1}\\ &\quad = (\Pi U^{\times}(\tau)^{-1} + (I-L)\,U(\tau)^{-1})^{-1}\,(I-L)\,U(\tau)^{-1}\\ &\quad = U^{\times}(\tau)(\Pi + (I-L)\,U(\tau)^{-1}\,U^{\times}(\tau))^{-1}\,(I-L)\,U(\tau)^{-1}. \end{aligned}$$

Then it follows from the boundedness of (41) that (48) holds. We conclude from Theorem 2.2 that indeed $\rho_\tau(t)$ converges in L_p to $\rho(t)$. Now remark that

$$\phi_\tau(t) = C(t)\,\rho_\tau(t) + B(t)\,f(t), \qquad \phi(t) = C(t)\,\rho(t) + B(t)\,f(t),$$

and conclude that ϕ_τ converges in L_p to ϕ. ∎

Notice that in general the boundary condition $(I-L)\,U(\tau)^{-1}\,\rho(\tau) = 0$ in (47) is different from the boundary condition $(I-\Pi)\,U^{\times}(\tau)^{-1}\,\rho(\tau) = 0$, which would be the boundary condition suggested by Remark 2.4 for this problem.

As for the case of the differential equations, here we also could consider the equation on the finite interval with a right hand side that depends on τ. Instead of using Theorem 2.2, we would then apply Theorem 2.3 to obtain convergence in L_p of the solution of the equation on the finite interval to the solution of the half line equation. We omit the details for this generalization.

There is another way to prove Theorem 5.1. To see this, let us consider the operator K_τ on $L_p^m[0, \tau]$ defined by

$$(K_\tau \phi)(t) = \phi(t) + \int_0^\tau k(t, s)\, \phi(s)\, ds,$$

where $k(t, s)$ is given by (38). Condition (41) and the argument used in the proof of Theorem 2.3 show that the operators K_τ^{-1} are uniformly bounded in the operator norm. Hence by the general theory of the projection method (see [5, Theorem II.2.1]) it follows that the finite section method converges. Notice that the operator K_τ^{-1} corresponds to the operator T_τ appearing in the proof of Theorem 2.3.

5.2. *The Time-Invariant Case*

In this section we consider the integral equation (37) with kernel

$$k(t, s) = \begin{cases} Ce^{tA} L e^{-sA} B, & 0 \leqslant s < t < \infty, \\ -Ce^{tA}(I - L)\, e^{-sA} B, & 0 \leqslant t < s < \infty, \end{cases} \tag{49}$$

where A is an $n \times n$ matrix with no eigenvalues on the imaginary axis, B and C are matrices of sizes $n \times m$ and $m \times n$, respectively, and L is a projection of $\mathbb{C}^n$ onto the space spanned by the eigenvectors and generalized eigenvectors of A corresponding to the eigenvalues in the left half plane. Recall that this means that L is an exponential dichotomy of e^{tA}. We have the following theorem.

THEOREM 5.2. *Let $k(t, s)$ be given by* (49). *Assume that the matrix $A^\times = A - BC$ has no eigenvalues on the imaginary axis, and let $L^\times$ be the spectral projection of $A^\times$ with respect to the left half plane. If $LL^\times + (I - L) \times (I - L^\times)$ is invertible, then the finite section method converges for* (37).

Proof. First note that if $LL^\times + (I - L)(I - L^\times)$ is invertible, then $\operatorname{Im} L^\times \oplus \operatorname{Ker} L = \mathbb{C}^n$. Hence, it follows (see the first part of the proof of Theorem 5.1) that Eq. (37) has a unique solution in $L_p^m[0, \infty)$. According to [8, Theorem II.2.2] (cf. the proof of Theorem 5.1), this unique solution of (37), say ϕ, can be expressed as $\phi(t) = -C\rho(t) + f(t)$, where $\rho(t)$ is the solution in $L_p^n[0, \infty)$ of the differential equation with boundary conditions

$$\dot{\rho}(t) = A^\times \rho(t) + Bf(t), \qquad 0 \leqslant t < \infty,$$

$$L\rho(0) = 0.$$

According to [7, Theorem 2.1] (cf. the proof of Theorem 5.1), the solution of (40) is $\phi_\tau(t) = -C\rho_\tau(t) + Bf(t)$, where $\rho_\tau(t)$ is the solution of

$$\dot{\rho}(t) = A^\times \rho(t) + Bf_\tau(t), \qquad 0 \leqslant t \leqslant \tau,$$
$$L\rho(0) + (I-L)\,\mathrm{e}^{-\tau A}\rho(\tau) = 0.$$

So it will be sufficient to show that ρ_τ converges in L_p to ρ. Notice that $\operatorname{Im} L$ is invariant under A. It follows that $(I-L)\,\mathrm{e}^{-\tau A}\rho(\tau) = 0$ is equivalent to $(I-L)\,\rho(\tau) = 0$, and therefore ρ_τ is a solution of

$$\dot{\rho}(t) = A^\times \rho(t) + Bf_\tau(t), \qquad 0 \leqslant t \leqslant \tau,$$
$$L\rho(0) + (I-L)\,\rho(\tau) = 0$$

Moreover, $\operatorname{Ker} L \oplus \operatorname{Im} L^\times = \mathbb{C}^n$, and hence we can apply Theorem 3.2 with $Q = L$ and $L_A = L^\times$ to obtain that ρ_τ converges in L_p to ρ. ▮

It is known, see [3, Sect. 5], that in the special case where the projection L is the spectral projection of A corresponding to the eigenvalues in the left half plane, the invertibility of $LL^\times + (I-L)(I-L^\times)$ is equivalent to the condition that the finite section method converges for (37). In the general case we need an extra condition to prove the converse of Theorem 5.2. In fact we will require that the spectral projection L_A of A with respect to left half plane is such that $\operatorname{Im} L^\times \oplus \operatorname{Ker} L_A = \mathbb{C}^n$. According to [2, Theorem 3.4] this requirement is equivalent to the unique solvability of Eq. (37) for the case when $L = L_A$.

THEOREM 5.3. *Let $k(t, s)$ be given by* (49). *Assume that the matrix $A^\times = A - BC$ has no eigenvalues on the imaginary axis, and let $L^\times$ be the spectral projection of $A^\times$ with respect to the left half plane. Let L_A be the spectral projection of A with respect to the left half plane. Assume that* $\operatorname{Im} L^\times \oplus \operatorname{Ker} L_A = \mathbb{C}^n$ *and that Eq.* (37) *is uniquely solvable for each right hand side $f \in L_p^m[0, \infty)$. Then the finite section method converges for* (37) *if and only if $LL^\times + (I-L)(I-L^\times)$ is invertible.*

Proof. It follows from Theorem 5.2 that if $LL^\times + (I-L)(I-L^\times)$ is invertible, then the finite section method converges for (37).

Conversely, assume that the finite section method converges for (37). We will prove that $LL^\times + (I-L)(I-L^\times)$ is invertible. Let $T_A \colon L_p^m[0, \infty) \to L_p^m[0, \infty)$ be the linear operator given by

$$(T_A \phi)(t) = \phi(t) + \int_0^\infty k_A(t, s)\,\phi(s)\,ds, \qquad 0 \leqslant t < \infty,$$

where

$$k_A(t,s)=\begin{cases} Ce^{tA}L_A e^{-sA}B, & 0\leqslant s<t<\infty,\\ -Ce^{tA}(I-L_A)\,e^{-sA}B, & 0\leqslant t<s<\infty.\end{cases}$$

Since $\operatorname{Im} L^\times \oplus \operatorname{Ker} L_A=\mathbb{C}^n$, it follows that T_A is invertible. Let $T\colon L_p^m[0,\infty)\to L_p^m[0,\infty)$ be given by

$$(T\phi)(t)=\phi(t)+\int_0^\infty k(t,s)\,\phi(s)\,ds, \qquad 0\leqslant t<\infty.$$

Then

$$((T-T_A)\,\phi)(t)=\int_0^\infty Ce^{tA}(L-L_A)\,e^{-sA}B\phi(s)\,ds.$$

Hence the operator $T-T_A$ is a finite rank operator. By assumption the finite section method converges for T. Therefore (see [5, Theorem II.3.1], also [1, Theorem 4.4]), the finite section method also converges for the compact perturbation T_A of T. According to [3, Sect. 5], this implies that $\operatorname{Im} L_A\oplus \operatorname{Ker} L^\times=\mathbb{C}^n$. Since we know that $\operatorname{Im} L=\operatorname{Im} L_A$, we may conclude that

$$\operatorname{Im} L\oplus \operatorname{Ker} L^\times=\mathbb{C}^n, \qquad \operatorname{Ker} L\oplus \operatorname{Im} L^\times=\mathbb{C}^n.$$

The second of these equalities follows from the assumed invertibility of T [8, Theorem II.7.4]. Both these equalities together are equivalent to the invertibility of $LL^\times+(I-L)(I-L^\times)$ (cf. [3, Sect. 5]). ∎

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